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A note about teleparallel supergravity

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Abstract. The supersymmetric extension of teleparallel gravity is discussed. It is found that teleparallel supergravity is equivalent to the usual supergravity when the vierbein, the gravitino field and the connection are not identified with the component of the gauge potential, but are obtained taking the limit $m \to 0$ in (81), (83) and (84) of P. Salgado, S. del Campo, M. Cataldo, Phys. Rev. D 68, 024021 (2003). It is also shown that the successful formalism of Grignani–Nardelli permits one to obtain teleparallel-supergravity in $(2+1)$ dimensions from $(3+1)$-dimensional teleparallel supergravity.

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1 Introduction

Teleparallel gravity is a gauge theory for the translation group [1] whose generators satisfy the Lie algebra

$$[P_A, P_B] = 0. \tag{1}$$

The Yang–Mills connection for this group is given by

$$A = e^A P_A, \tag{2}$$

where the e^A are the corresponding gauge field.

The exterior covariant derivative is then given by

$$D\phi = (d + A)\phi = (d + e^A P_A)\phi, \tag{3}$$

i.e.,

$$\begin{aligned} D_\mu\phi &= (\partial_\mu + e^A_\mu P_A)\phi = (\delta^A_\mu + e^A_\mu)\partial_A\phi \\ &= \left(\partial_\mu x^A + e^A_\mu\right)\partial_A\phi = \widetilde{e}^A_\mu\partial_A\phi, \end{aligned} \tag{4}$$

where the $\widetilde{e}^A$ define a new tetrad which is invariant under translations.

The torsion 2-form is given by

$$\widetilde{T}^A = d\widetilde{e}^A, \tag{5}$$

which in terms of the so called contorsion tensor $\widetilde{K}^{AB}$ takes the form [2]

$$\widetilde{T}^A = \widetilde{K}^A_C\widetilde{e}^C. \tag{6}$$

The $(3+1)$-dimensional action for teleparallel gravity given by (49) of [3], can then be written in the form

$$S_{\mathrm{Tel}} = -\int \varepsilon_{ABCD}\widetilde{K}^A_L\widetilde{K}^{LB}\widetilde{e}^C\widetilde{e}^D, \tag{7}$$

where $\widetilde{e}^A$ is given by [3, 4]

$$\widetilde{e}^A = Dx^A = dx^A + e^A, \tag{8}$$

which is a gauge invariant. In fact, since under translations x^A and the gauge potential e^A change as

$$\delta x^A = -\rho^A; \quad \delta e^A = d\rho^A, \tag{9}$$

we have

$$\delta\widetilde{e}^A = 0. \tag{10}$$

Using the identity $D\varepsilon_{ABCD} = 0$ it is possible to show that the action (7) can be written, up to a boundary term, as

$$S_{\mathrm{Tel}} = -\int \varepsilon_{ABCD}\widetilde{K}^{AB}\widetilde{T}^C\widetilde{e}^D. \tag{11}$$

From (10) we can see that the action (11) is invariant under local translations and by construction it is invariant under local Lorentz rotations and under diffeomorphisms. This means that the action (11) is genuinely invariant under the Poincaré group and under diffeomorphisms. In the Einstein–Hilbert action, the tetrad is given by the gauge potential e^A while in teleparallel gravity the tetrad is defined as in (8). A consequence of this situation is that the action for teleparallel gravity is genuinely invariant under the Poincaré group and under diffeomorphisms whereas the Einstein–Hilbert action is invariant only under the Lorentz group and under diffeomorphisms. This means that the

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actions for teleparallel gravity and the Einstein–Hilbert action *are not* fully equivalent.

The purpose of this paper is to find a supersymmetric extension of teleparallel gravity and to show that teleparallel supergravity is equivalent to the usual supergravity when the vierbein, the gravitino field and the spin connection are not identified with the component of the gauge potential, but when they are obtained taking the limit $m \to 0$ in (81), (83) and (84) of [5] (see also [6]).

2 Supergravity

$D = 3 + 1$, $N = 1$ supergravity is based on the super-Poincaré algebra [7],

$$[P_A, P_B] = 0, \tag{12}$$

$$[J_{AB}, P_C] = \eta_{AC} P_B - \eta_{BC} P_C, \tag{13}$$

$$[J_{AB}, J_{CD}] = \eta_{AC} J_{BD} - \eta_{BC} J_{AD} + \eta_{BD} J_{AC} - \eta_{AD} J_{BC}, \tag{14}$$

$$[J_{AB}, Q^\alpha] = -\frac{1}{2} (\gamma_{AB})^\alpha_\beta Q^\beta, \tag{15}$$

$$[P_A, Q_\beta] = 0, \tag{16}$$

$$[Q^\alpha, Q_\beta] = \frac{1}{2} \left(\gamma^A\right)^\alpha_\beta P_A. \tag{17}$$

The connection for this group is given by

$$A = e^A P_A + \frac{1}{2} \omega^{AB} J_{AB} + \overline{Q} \psi, \tag{18}$$

whose components, the vierbein e^A, the spin connection ω^{AB}, and the Rarita–Schwinger field ψ transform under Poincaré translations as

$$\delta e^A = D\rho^A, \quad \delta\omega^{AB} = 0, \quad \delta\psi = 0, \tag{19}$$

and under Lorentz rotations as

$$\delta e^A = \kappa^A_B e^B, \quad \delta\omega^{AB} = -D\kappa^{AB}, \quad \delta\psi = \frac{1}{4} \kappa^{AB} \gamma_{AB} \psi. \tag{20}$$

Finally, under supersymmetry they transform as

$$\delta e^A = \frac{1}{2} \bar{\varepsilon} \gamma^A \psi, \quad \delta\omega^{AB} = 0, \quad \delta\psi = D\varepsilon. \tag{21}$$

The generalized Weyl lemma,

$$\mathcal{D} e^A_\mu = \partial_\nu e^A_\mu - \omega^A_{B\mu} e^B_\nu - \frac{1}{4} \bar{\psi}_\mu \gamma^A \psi_\nu - \Gamma^\lambda_{\mu\nu} e^A_\lambda = 0, \tag{22}$$

implies that the so called supertorsion is given by

$$\hat{T}^A = T^A - \frac{1}{2} \bar{\psi} \gamma^A \psi.$$

Supergravity is the theory of the gravitational field e^A interacting with a spin 3/2 Rarita Schwinger field ψ [8]. In the simplest case there is just one spin 3/2 Majorana fermion, usually called the gravitino. The theory is described by the action

$$S = \int \varepsilon_{ABCD} e^A e^B R^{CD} + 4\overline{\psi} \gamma_5 e^A \gamma_A D\psi, \tag{23}$$

which is invariant under diffeomorphisms, under local Lorentz rotations, but it is not invariant under Poincaré translations. In fact, under local Poincaré translations,

$$\delta S = 2 \int \varepsilon_{ABCD} R^{AB} \hat{T}^C \rho^D + \text{surf. term}. \tag{24}$$

The invariance of this action requires the vanishing of the supertorsion:

$$\hat{T}^A = 0. \tag{25}$$

The same condition is necessary for the invariance of the action under a supersymmetry transformation. This implies that the connection is no longer an independent variable. Rather, its variation is given in terms of δe^A and $\delta\psi$ and differs from the one dictated by group theory [5].

The condition $\hat{T}^A = 0$ leads to $\omega^{AB} = \omega^{AB}(e, \psi)$ which implies that the connection is no longer an independent variable, and its variation $\delta(\varepsilon)\,\omega^{AB}$ is given in terms of $\delta(\varepsilon)\,e^A$ and $\delta(\varepsilon\ )\,\psi$, which implies that, without the auxiliary fields, the gauge algebra does not close, as (10) of [7] shows. Therefore the condition $\hat{T}^A = 0$ not only breaks local Poincaré invariance, but also the supersymmetry transformations.

3 Teleparallel supergravity

The action for teleparallel gravity has been constructed from a tetrad given by $\widetilde{e}^A = Dx^A = dx^A + e^A$, which can be understood as the tetrad corresponding to the tetrad used in the construction of an action for gravity invariant under the Poincaré group [9]. This tetrad can be obtained taking the Poincaré limit of the AdS non-linear gauge potential given in (3.19) of [10]. This fact suggests that an action for teleparallel supergravity could be constructed from a vierbein and a gravitino field which corresponds to the vierbein, the gravitino field and the spin connection used in the construction of a supergravity invariant under the supersymmetric extension of the Poincaré group, which was obtained taking the limit $m \to 0$ in (81), (83) and (84) of [5].

The corresponding Yang–Mills connection is given by

$$A = \widetilde{e}^A P_A + \overline{\Psi} Q,$$

where [5]

$$\widetilde{e}^A = dx^A + e^A + \mathrm{i}\left(2\overline{\psi} + D\overline{\theta}\right)\gamma^A \theta,$$

$$\Psi = \psi + D\theta,$$

with $D\theta = d\theta + \frac{1}{2} K^{ab} \gamma_{ab} \theta$.

Since x^A, e^A, ψ, and θ transform, under translation, as

$$\delta x^A = -\rho^A, \quad \delta e^A = d\rho^A,$$

$$\delta\psi = 0, \quad \delta\theta = 0,$$

and under local supersymmetry transformations as

$$\delta x^A = -\mathrm{i}\bar{\varepsilon}\gamma^A\theta, \quad \delta e^A = -2\mathrm{i}\bar{\varepsilon}\gamma^A\psi,$$
$$\delta\psi = D\varepsilon, \quad \delta\theta = -\varepsilon,$$

we see that the action for teleparallel supergravity,

$$S_{\mathrm{TSG}} = -\int \varepsilon_{ABCD}\widetilde{K}^A_L\widetilde{K}^{LB}\widetilde{e}^C\widetilde{e}^D - 4\overline{\Psi}\gamma_5\gamma_d D\Psi\widetilde{e}^D, \quad (26)$$

or also

$$S_{\mathrm{TSG}} = -\int \varepsilon_{ABCD}\widetilde{K}^{AB}\widetilde{T}^C\widetilde{e}^D - 4\overline{\Psi}\gamma_5\gamma_d D\Psi\widetilde{e}^D + \text{surface term},$$

is invariant under local translations, under local supersymmetry transformations and by construction is invariant under local Lorentz rotations and under diffeomorphisms. This means that the action (26) is genuinely invariant under the Poincaré supergroup and under diffeomorphisms.

4 Dimensional reduction

The dimensional reduction process, as well as the notation, is similar to that used in [11,12]. Latin indices $a, b, c, \ldots = 0, 1, 2$ and capital latin indices $A, B, C, \ldots = 0, 1, 2, 3$ denote (2+1) and (3+1) internal (gauge) indices, respectively. They are raised and lowered by the Minkowski metrics

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (27)$$

and

$$\eta_{AB} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (28)$$

In the dimensional reduction the first three values of $A, B, C, \ldots$ will denote the corresponding $(2+1)$ internal indices $a, b, c, \ldots$, i.e. $A = (a, 3)$, $B = (b, 3)$, $C = (c, 3)$, ... We shall use the antisymmetric symbol ε^{ABCD} with $\varepsilon^{0123} = 1$ and in $(2+1)$ dimensions $\varepsilon^{abc} = \varepsilon^{abc3}$, so that $\varepsilon^{012} = 1$.

Following the procedure of [11] we carried out a dimensional reduction of the Poincaré generators of the $(3+1)$-dimensional theory and, correspondingly, of the space-time dimensions that, from the $(3+1)$-dimensional action (26) and the algebra (12)–(17), lead to the $(2+1)$-dimensional action. With such reductions from the (3+1) gauge transformations (19), (20) and (21), we shall obtain the corresponding gauge transformations in $(2+1)$ dimensions.

The dimensional reduction leading from the $(3+1)$-dimensional supergravity theory to the $(2+1)$-Chern–Simons supergravity theory is given in Table 1 [11].

Table 1. The dimensional reduction leading from the $(3+1)$-dimensional supergravity theory to the $(2+1)$-Chern–Simons supergravity theory

Dimensional reduction	
$(3+1)$ dimensions	$(2+1)$ dimensions
e^3	dx^3
e^a	e^a
K^{ab}	K^{ab}
K^{a3}	0
x^a	x^a
x^3	0
ρ^a	ρ^a
ρ^3	0
ψ	ψ
γ^{abc}	γ^{abc}
γ^3	0

Here the γ' with multiple indices are antisymmetrized products of gamma matrices, which for d dimensions satisfy the relation [13]

$$\gamma^{i_1 i_2 \ldots i_k} = \alpha\varepsilon^{i_1 i_2 \ldots i_d}\gamma_{i_{k+1}\ldots i_d}\gamma^{d+1}, \quad (29)$$

with

$$\alpha = \frac{1}{(d-k)}(-1)^{\frac{1}{2}k(k-1)+\frac{1}{2}d(d-1)}. \quad (30)$$

It is straightforward to verify that the $(3+1)$-gauge transformations (19), (20) and (21), with the identifications of Table 1 of dimensional reduction are mapped onto

$$\delta x^a = -\rho^a, \ \delta e^a = d\rho^a, \ \delta\theta = 0, \ \delta\psi = 0, \quad (31)$$

$$\delta x^a = -\mathrm{i}\bar{\varepsilon}\gamma^a\theta, \ \delta e^a = -2\mathrm{i}\bar{\varepsilon}\gamma^a\psi, \ \delta\theta = -\varepsilon, \ \delta\psi = D\varepsilon, \quad (32)$$

i.e., onto the correct $(2+1)$-dimensional gauge transformations. In particular, the quantities that are set to a constant in Table 1 consistently have vanishing gauge transformations. In the same way we have

$$R^{AB} = \begin{pmatrix} K^{ab} & K^{a3} \\ K^{3b} & K^{33} \end{pmatrix} = \begin{pmatrix} K^{ab} & 0 \\ 0 & 0 \end{pmatrix}, \quad (33)$$

$$\widetilde{e}^A = \begin{pmatrix} \widetilde{e}^a \\ \widetilde{e}^3 \end{pmatrix} = \begin{pmatrix} e^a + dx^a + \mathrm{i}(2\overline{\psi} + D\overline{\theta})\gamma^a\chi \\ dx^3 \end{pmatrix}, \quad (34)$$

$$\Psi = \psi + D\theta. \quad (35)$$

From (29) we see that, for $d = 4$ and $k = 3$,

$$\gamma^{ABC} = -\varepsilon^{ABCD}\gamma_D\gamma^5, \quad (36)$$

which allows one to write the action for $(3+1)$-dimensional teleparallel supergravity in the form

$$S_{\mathrm{TSG}} = -\int \varepsilon_{ABCD}\widetilde{K}^A_L\widetilde{K}^{LB}\widetilde{e}^C\widetilde{e}^D - 4\overline{\Psi}\gamma_5\gamma_d D\Psi\widetilde{e}^D, \quad (37)$$

$$S^{4D}_{TSG} = -\int \varepsilon_{ABCD}\left(\widetilde{K}^A_L \widetilde{K}^{LB}\widetilde{e}^C\widetilde{e}^D - \frac{1}{3!}\overline{\Psi}\gamma^{ABC}\widetilde{e}^D D\Psi\right).$$

By substituting the content of Table 1 of dimensional reduction and (33) and (34) into the action (37) one gets

$$S^{4D}_{TSG} = -\int\left(2\varepsilon_{abc3}\widetilde{K}^a_l\widetilde{K}^{lb}\widetilde{e}^c - \frac{4}{3!}\varepsilon_{abc3}\overline{\Psi}\gamma^{abc}D\Psi\right)dz^3. \tag{38}$$

Using (34) and (35) and the identity $\gamma_{ab} = -\mathrm{i}\varepsilon_{abc}\gamma^c$ we find that the first term is

$$\begin{aligned} &2\varepsilon_{abc}\widetilde{K}^a_l\widetilde{K}^{lb}\widetilde{e}^c \\ &= 2\varepsilon_{abc}\widetilde{K}^a_l\widetilde{K}^{lb}\widetilde{e}^c + 2\varepsilon_{abc}\left(d\widetilde{K}^{ab} - \widetilde{K}^a_l\widetilde{K}^{lb}\right)Dx^c \\ &\quad -4\left(d\widetilde{K}^{ab} - \widetilde{K}^a_l\widetilde{K}^{lb}\right)\overline{\psi}\gamma_{ab}\theta \\ &\quad -2\left(d\widetilde{K}^{ab} - \widetilde{K}^a_l\widetilde{K}^{lb}\right)(D\overline{\theta})\gamma_{ab}\theta. \end{aligned} \tag{39}$$

Using (29), $\gamma^{abc} = -\varepsilon^{abc}I$, and the identities

$$DD\theta = -\frac{1}{2}\left(d\widetilde{K}^{ab} - \widetilde{K}^a_l\widetilde{K}^{lb}\right)\gamma_{ab}\theta,$$
$$\overline{\theta}\gamma_{ab}\psi = -\overline{\psi}\gamma_{ab}\theta,$$

we find that the second term can be written as

$$\begin{aligned} &\frac{4}{3!}\varepsilon_{abc3}\overline{\Psi}\gamma^{abc}D\Psi \\ &= \frac{4}{3!}\varepsilon_{abc3}\overline{\psi}\gamma^{abc}D\psi + 4D(\overline{\theta}D\psi) \\ &\quad -4\left(d\widetilde{K}^{ab} - \widetilde{K}^a_l\widetilde{K}^{lb}\right)\overline{\psi}\gamma_{ab}\theta \\ &\quad -2(D\overline{\theta})\left(d\widetilde{K}^{ab} - \widetilde{K}^a_l\widetilde{K}^{lb}\right)\gamma_{ab}\theta. \end{aligned} \tag{40}$$

By substituting (39) and (40) in (38) we obtain

$$\begin{aligned} S^{4D} = -\int\Big\{&2\varepsilon_{abc}\widetilde{K}^a_l\widetilde{K}^{lb}e^c - \frac{4}{3!}\varepsilon_{abc}\overline{\psi}\gamma^{abc}D\psi \\ &+2\varepsilon_{abc}\left(d\widetilde{K}^{ab} - \widetilde{K}^a_l\widetilde{K}^{lb}\right)Dx^c + 4D(\overline{\theta}D\psi)\Big\}. \end{aligned} \tag{41}$$

Using the Bianchi identity

$$D\left(d\widetilde{K}^{ab} - \widetilde{K}^a_l\widetilde{K}^{lb}\right) = 0,$$

$\varepsilon_{abc}\varepsilon^{abc} = -3!$ and (29) with $d = 3$ and $k = 3$, we find that the action for (2 + 1)- supergravity is given by

$$S^{4D} \to S^{3D} = -\int \varepsilon_{abc}\widetilde{K}^a_l\widetilde{K}^{lb}e^c + 4\overline{\psi}D\psi + \text{ surface term.} \tag{42}$$

This result proves that the dimensional reduction from (3 + 1)-dimensional teleparallel supergravity to (2 + 1) teleparallel supergravity is possible.

5 Comments

We have found a supersymmetric extension for teleparallel gravity and we have shown that teleparallel supergravity is equivalent to the usual supergravity when the vierbein, the gravitino field and the spin connection are not identified with the component of the gauge potential, but obtained taking the limit $m \to 0$ in (81), (83) and (84) of [5].

We have also shown that the successful formalism proposed in [11, 12] permits one to obtain teleparallel supergravity in (2 + 1) dimensions from (3 + 1)-dimensional teleparallel supergravity.

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